**Relaxation Time Approximation**

So going to look at using the Boltzman equation to model the interaction of particles with some sort of impurity potential randomly distributed throughout the interior of our substance and also possible collisions between particles. We’ll presume there is some driving field present which forces the particles to accelerate, and then the collisions with impurities/other particles scatter the particles into different states, serving to try to force them back to an equilibrium arrangement. So let’s go back to:



And instead of actually solving this equation, we’re going to write a different equation that captures its spirit, but is easier.

**(Re)Deriving LHS of equation**

In the Relaxation Time Approximation, we just replace the entire RHS with the expression (fleq – f)/τ, where τ is the mean scattering time. Before we get to that, and to motivate what follows, let’s neglect the RHS for the time being. And let’s rederive the LHS of the single-particle distribution function equation. So suppose that at time t0 we have a single particle distribution of particles, f0(**r**,**k**,t0­). If the system is in thermodynamic equilibrium, then the distribution would be something like the equilibrium distribution:



Now suppose we switch on an additional single particle force, Fnc(r), say, so that the total single particle force is F = -∇φ + Fnc (I suppose the extra force Fnc doesn’t have to be non-conservative as indicated, but it typically is that sort of thing). We assume in our classical picture that the particles in the presence of some field change their state according to:



Then let us evolve our distribution function through time. Consider two times: t, and t – dt. f(**r**,**k**,t) will depend on f(**r**,**k**,t-dt) in that all the states at (**r**,**k**) at time t must have gotten there from (**r**’,**k**’) at time t – dt. (**r**’,**k**’) is just the state (**r**,**k**) evolved backwards in time by an amount dt. So,



which gives us our distribution function evolution equation again, just like we had above (though in the single particle case here)



So this is the equation for the distribution function assuming all the (r,k) states evolve according to H on their own – no scattering.

**Replacing RHS of equation with relaxation time approximation**

Now let’s incorporate the scattering potential. We suppose that all the RHS potential(s) does is to scatter the particles from state rk to r´k´ and vice versa. And so we could start by postulating,



where by density I mean per unit volume per unit momentum. Then we introduce a phenomenological parameter – the scattering/relaxation time – defined as follows…So suppose the probability per particle of scattering out of (r,k) in an interval dt is:



Then the total probability density of particles scattering out of (r,k) in time interval about t is:



And so the probability density rate of scattering out of (r,k) is:



The probability density rate incomming is determined in the following way. We suppose we’re in local equilibrium. And we can allow a time-dependent equilibrium state, I suppose, as long as it varies much slower than the mean free collision time. Then we might have, where β = 1/T, **u** is the average translational velocity, and μ is the chemical potential:



or some such, but no perturbative fields, etc. We denote this local equilibrium distribution fleq(**r**,**k**,t). Note β(r) is a position-dependent temperature, u(r) is a local convective velocity, μ(r) is the local chemical potential. φ(r) is the equilibrium potential energy, if present. Do note that if we have impurity scattering, then there would be no local convective velocity term **u**(r) in the local equilibrium fleq expression. The form of this expression will receive more support when we get to the 1PI and 2PI Boltzman equation. Basically, fleq(r,k,t) is the expression which annihilates the RHS of the 1PI and 2PI Boltzman equation.

So then the probability density coming into rk must equal the # leaving. The probability density leaving is, according to the above, fleq(**r**,**k**,t)dt/τ(**r**,**k**). And so we can say:



Now here’s the thing. Collisions are the mechanism for ‘maintaining’ local thermal equilibrium. And we assume that even when we turn on the perturbative fields, the *distribution* of particles leaving any given scattering site (i.e., how many into r1k1, how many into r2k2, etc.) is unchanged. If this is true, then the total influx of particles into r1k1, r2k2, etc., which would just be the sum of all influxes into r1k1, r2k2, etc., from all other scattering sites, would likewise remain unchanged. And so, repeating myself, the distribution of particles scattering *into* rk is unchanged. And so we can say,



Again, the probability it will scatter at (r,k) depends on f(r,k,t). But where it will go to next, at t + dt, doesn’t depend on f(r,k,t), only on the local equilibrium distribution, and so the collisions erase all memory of the distribution prior to the collision. So our equation is modified to:



In a while we will write the RHS as the rate of collisions occurring in the material. So interpreting our formula in that manner we see that the rate of collisions occurring in the material is proportional to the deviation of the system from TD equilibrium. So the collisions are the mechanism for returning the system to TD equilibrium. Harping on the relaxation time theme again. Suppose we inject an electron into a metal w/ an initial momentum above the Fermi surface. As soon as it enters the metal, it will be subject to its Hamiltonian, and will soon relax into the thermal equilibrium state via a bunch of scattering events, until its momentum resides on the Fermi surface. If we injected a bunch of electrons into the metal w/ momenta, say, above kF. Then this would be a non-thermal equilibrium situation, as above. This is obviously the case since the distribution of electrons’ energies in the metal won’t initially be equal to nF(εp). But via a bunch of scattering events, the electrons injected, and the others, will rearrange themselves into a new Fermi sphere (if at T = 0), or into the nF appropriate to whatever the T is. Thus, it is truly the collisions, or whatever scattering events there are, which are responsible for reestablishing equilibrium.

**Semi-classical RTA**

How will things change if we incorporate a little quantum mechanics? So the formalism above is built up from classical physics. But we can/will incorporate some quantum mechanics by making a so-called semi-classical approximation. We will assume that the quantum particles consist of wavepacket superpositions which give them well defined positions and momenta, and spin. So we’ll presume something like fσ(**r**,**k**,t). And we’ll presume that these variables evolve according to classical mechanics. So how does the LHS of our equation change? Well we still have:



Then we evolve our distribution in time, as before. So consider two times: t, and t – dt. fσ(**r**,**k**,t) will depend on fσ(**r**,**k**,t-dt) in that all the states at (**r**,**k**) at time t must have gotten there from (**r**’,**k**’) at time t – dt. (**r**’,**k**’) is just the state (**r**,**k**) evolved backwards in time by an amount dt. So,



which gives us:



And then incorporating scattering, as before, gives us:



where now we allow the scattering rate to depend on spin. But now apropos the RHS, what is fleq,σ? So we just want to update our classical expression expression:



by incorporating quantum mechanics. Well an equilibrium distribution function would be something like,



where n labels the quantum state, |n>, of the particle. In the semi-classical approximation we assume that the states are particle-like superpositions of quantum waves |n> = Σk’ck´(r)|k´σ> with a well-defined position r, and momentum k, and spin σ. You could call them coherent states I guess, and specify them as |rkσ>. The energies could then be treated classically, assuming the particle is localized on a scale smaller than the potential varies. So we say, εn → ε(r,k) = |k-mu(r)|2/2m + φ(r). We can could add a spin part too if desired. But I’m not going to bother. If the energy is allowed a spatial dependence, then we should allow a spatial dependence to the temperature: β(r), and chemical potential μ(r) as well. Now we also need to convert our discrete probability distribution fn into a continuous one fσ(**r**,**p**). To do that we have to divide by the width of the state in phase space. Most typically, we’re in a some container with volume V, and periodic boundary conditions. Periodic boundary conditions discretize the allowed **p** = ℏ**k** values: kx = 2πnx/Lx, ky = 2πny/Ly, kz = 2πnz/Lz. So we have the spacings Δpx = 2πℏ/Lx, Δpy = 2πℏ/Ly, Δpz = 2πℏ/Lz. And so the volume of phase space a state occupies is Δ3p = ΔpxΔpyΔpz = (2πℏ)3/LxLyLz = (2πℏ)3/V. And for plane waves, kind of belieing my claim that the states are somewhat localized, we can take Δ3r to be V. So then the volume of phase space occupied by a our semi-classical state (r,p) would be: Δ3rΔ3p = V·(2πℏ)3/V = (2πℏ)3. It’d be more rigorous to say that Δ3r is much smaller than V, and Δ3k is correspondingly much larger than (2π)3/V so that there can be a superposition of plane waves, but leaving the product the same. Any way we’d have:



This is the customary pre-factor we’re used to seeing associated with the classical distribution function. One more note: I’ll probably leave off the ℏ factor a lot. Another way we could do it is:



Let’s just say that |n> = |kσ>. Then,



Then, to make progress we need to convert the Dirac delta to a Kronecker delta (remember these are 3D deltas). Well we can say,



So filling this in, we have:



Then if fpσ has an equilibrium distribution function, we get the same form as surmised above, if we generously allow ex post facto for spatial variation. And I guess I’ll mention a cosmetic definition. I’m going to keep the definition:



as the number of particles in a state with specified semi-classical energy. So in terms of this, we may clearly write f as:



If we don’t care about spin, and just want to know how the distribution evolves w/r to space and momentum, then we can do,



And if we want to know just the distribution w/r to space, then we can do,



**Putting Boltzman equation in terms of semi-classical occupation numbers**

We can put the RTA equation directly in terms of occupation numbers [per Δ3rΔ3p = (2πℏ)3] given by



instead of probabilities. It has the properties,



using Δ3k = (2π)3/V. And averages would be done via:



We can put our RTA equation in terms of nkσ(r). Multiply both sides of our RTA equation by (2π)3N, and we get:



the same equation of course, he he. Here’s some more on the relationship between f and n,



where Δ3rΔ3p = (2πℏ)3. Last line is kind of fishy.